1. $f(x,y) = xy^2 - xy = x(y^2 - y)$

(a)

$f_x = y^2 - y \qquad f_x = 0 \Rightarrow y = 0, 1$

$f_y = x(2y-1) \qquad f_y = 0$

$f_{xx} = 0 \qquad \hookrightarrow y = 0 \Rightarrow x(-1) = 0 \Rightarrow x = 0$

$f_{xy} = 2y - 1 \qquad \hookrightarrow y = 1 \Rightarrow x(1) = 0 \Rightarrow x = 0$

$f_{yy} = 2x \qquad \Rightarrow \underline{(0,0)}, \underline{(0,1)}$

$\Rightarrow D = f_{xx} f_{yy} - f_{xy}^2 = 0(2x) - (2y-1)^2$

$= -(2y-1)^2$

$D(0,0) = -(-1)^2 = -1 < 0 \Rightarrow$ $\boxed{\text{saddle at } (0,0)}$

$D(0,1) = -(1)^2 = -1 < 0 \Rightarrow$ $\boxed{\text{saddle at } (0,1)}$

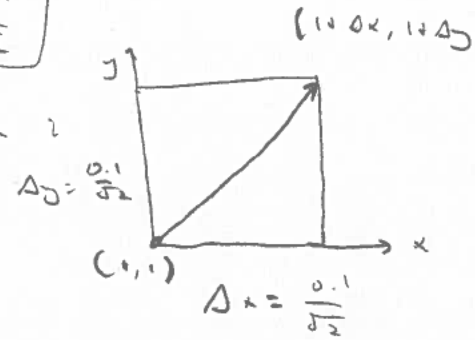
(b) $\dfrac{df}{ds} = \nabla f \cdot \vec{u} \qquad \vec{u} = \dfrac{\langle 1, 1 \rangle}{\sqrt{1^2 + 1^2}} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

$\nabla f = \langle y^2 - y, x(2y-1) \rangle$

$\Rightarrow \Delta f \approx [\nabla f(1,1) \cdot \vec{u}]\, \Delta s = \langle 0, 1(1) \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \cdot 0.1$

$= \boxed{\dfrac{0.1}{\sqrt{2}}}$

(c) Region:



$\left.\begin{array}{l} f_{xx} = 0 \\ f_{xy} = 2y - 1 \\ f_{yy} = 2x \end{array}\right\} \max = f_{yy}\Big|_{(1+\Delta x, 1+\Delta y)}$

$= 2\left(1 + \frac{0.1}{\sqrt{2}}\right)$

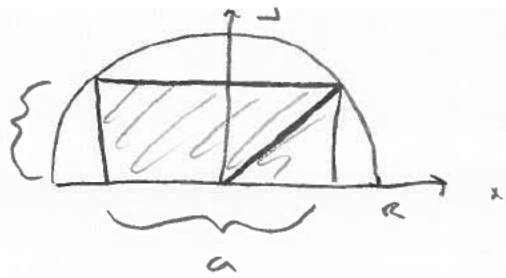
$= 2 + \sqrt{2}(0.1)$

$\Rightarrow E \leq \dfrac{M}{2}(\Delta x + \Delta y)^2 = \dfrac{2\left(1 + \frac{0.1}{\sqrt{2}}\right)}{2}\left(\frac{0.1}{\sqrt{2}} + \frac{0.1}{\sqrt{2}}\right)^2$

$= \left(1 + \frac{0.1}{\sqrt{2}}\right)\left(\sqrt{2}(0.1)\right)^2 = \left(1 + \frac{0.1}{\sqrt{2}}\right)(2(0.01))$

$= \boxed{\left(1 + \frac{0.1}{\sqrt{2}}\right)(0.02)}$

(2)

(a) 

Max area $a \cdot b$ subject to

$$\left(\frac{a}{2}\right)^2 + b^2 = R^2$$

$$\Rightarrow f(a,b) = ab \qquad \nabla f = \langle b, a \rangle$$

$$g(a,b) = \frac{a^2}{4} + b^2 \qquad \nabla g = \langle \frac{a}{2}, 2b \rangle \Rightarrow \boxed{\begin{array}{ll} b = \lambda \frac{a}{2} & (1) \\ a = \lambda 2b & (2) \\ \frac{a^2}{4} + b^2 = R^2 & (3) \end{array}}$$

<u>Note</u>: $a, b = 0$ gives area of $0 \Rightarrow$ <u>Not</u> a max, so we can divide by a, b

$$(2) \Rightarrow a = \lambda 2b \Rightarrow \lambda = \frac{a}{2b} \Rightarrow (1)\ b = \frac{a}{2b}\left(\frac{a}{2}\right) \Rightarrow b^2 = \frac{a^2}{4}$$

$$\Rightarrow b^2 + b^2 = R^2 \Rightarrow b^2 = \frac{R^2}{2} \Rightarrow \boxed{b = \frac{R}{\sqrt{2}}}$$

$$a^2 = 4b^2 \Rightarrow a = 2b \Rightarrow \boxed{a = \sqrt{2} R}$$

$$\left(\text{Check: } \frac{a^2}{4} + b^2 = \frac{2R^2}{4} + \frac{R^2}{2} = \frac{R^2}{2} + \frac{R^2}{2} = R^2 \checkmark\right)$$

(b) S[illegible] Area $= f(a,b) = f\left(\sqrt{2}R, \frac{R}{\sqrt{2}}\right)$

$$= (\sqrt{2}R)\left(\frac{R}{\sqrt{2}}\right) = \boxed{R^2}$$

3) (a) $u = x + 2y$ $\Rightarrow$ $u - v = (x + 2y) - (x - y)$

$v = x - y$

$= x + 2y - x + y = 3y$

$$\Rightarrow \boxed{y = \frac{u - v}{3}}$$

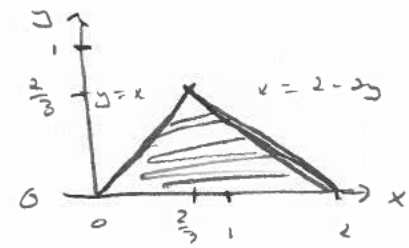
$u + 2v = (x + 2y) + 2(x - y)$

$= x + 2y + 2x - 2y = 3x$

$$\Rightarrow \boxed{x = \frac{u + 2v}{3}}$$

(b) $x = y$, $x = 2 - 2y$, $y = 0$, $y = \frac{2}{3}$

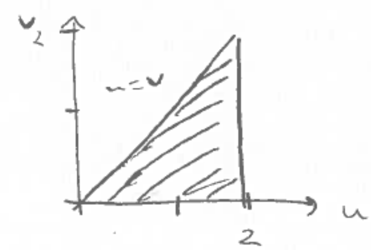
$y = 2 - 2y \Rightarrow 3y = 2 \Rightarrow y = \frac{2}{3}$ $\Rightarrow$



$u = x + 2y$ $\quad$ $x = y \Rightarrow x - y = 0 \Rightarrow v = 0$

$v = x - y$ $\quad$ $x = 2 - 2y \Rightarrow x + 2y = 2 \Rightarrow u = 2$

$y = 0 \Rightarrow u = x,\ v = x \Rightarrow u = v$ $\Rightarrow$



(c)

$(x + 2y)(x - y)^2 = uv^2$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3} \qquad |J| = \frac{1}{3}$$

(d)

$$\Rightarrow \boxed{\int_0^2 \int_0^u uv^2 \left(\frac{1}{3}\right) dv\, du} = \frac{1}{3} \int_0^2 u \frac{v^3}{3} \Big|_0^u du = \frac{1}{9} \int_0^2 u^4\, du$$

$$= \frac{1}{9} \frac{u^5}{5} \Big|_0^2 = \boxed{\frac{32}{45}}$$

(9) (a) $\nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 2xyz + 5z & e^x \cos yz & x^2 y \end{vmatrix}$

$= \vec{i}\left(x^2 + y e^x \sin(yz)\right) - \vec{j}\left(2xy - 2xy - 5\right) + \vec{k}\left(e^x \cos(yz) - 2xz\right)$

$= \langle x^2 + y e^x \sin(yz),\ 5,\ e^x \cos(yz) - 2xz \rangle \neq \vec{0}$

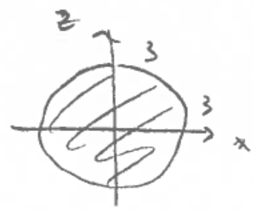
$\Rightarrow$ Not conservative

(b) Using Stokes we can deform the surface to any surface w/ the same boundary:

Boundary: $y = 1 \Rightarrow 1 = 10 - x^2 - z^2 \Rightarrow x^2 + z^2 = 9$

$\Rightarrow$ let $\underbrace{y = 1}_{\text{New surface}} \Rightarrow f(x,y,z) = y \Rightarrow \nabla f = \langle 0, 1, 0 \rangle$

Project onto xz plane: R:



, $\vec{p} = \vec{j}$

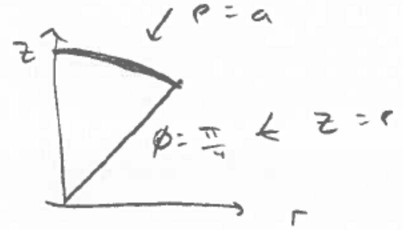
$\Rightarrow \iint_S (\nabla \times F) \cdot \vec{n}\, d\sigma = \iint_R \frac{(\nabla \times F) \cdot \nabla f}{\nabla f \cdot \vec{p}}\, dA$

$= \int_0^{2\pi} \int_0^3 \frac{5}{1}\, r\, dr\, d\theta = 5\left(\int_0^3 r\, dr\right)\left(\int_0^{2\pi} d\theta\right)$

$= 5\left(\frac{r^2}{2}\Big|_0^3\right)(2\pi) = \boxed{45\pi}$

(5)

(a)



$S_1: \rho = a \Rightarrow \sqrt{x^2+y^2+z^2} = a$

$\Rightarrow x^2+y^2+z^2 = a^2$

$S_2: z = r \Rightarrow z = \sqrt{x^2+y^2} \Rightarrow z^2 = x^2+y^2$

Intersection: $x^2+y^2+z^2 = a^2 \Rightarrow 2z^2 = a^2 \Rightarrow z = \frac{a}{\sqrt{2}}$

$\Rightarrow x^2+y^2 = \frac{a^2}{2}$

$f_1 = x^2+y^2+z^2 \Rightarrow \nabla f_1 = \langle 2x, 2y, 2z \rangle$

$f_2 = x^2+y^2-z^2 \Rightarrow \nabla f_2 = \langle 2x, 2y, -2z \rangle$

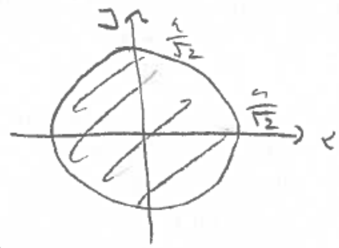
Fluxes: $F \cdot \nabla f_1 = 2x^2+2y^2+2z^2 = 2(x^2+y^2+z^2) = 2a^2$

$F \cdot \nabla f_2 = 2x^2+2y^2-2z^2 = 2(x^2+y^2) - 2(x^2+y^2) = 0$

↳ No flux through S_2!

$\Rightarrow$ Total Flux = Flux through S_1

Project onto xy-axes:



$, \vec{p} = \hat{z}$

$\nabla f_1 \cdot \vec{p} = 2z = 2\sqrt{a^2-x^2-y^2} = 2\sqrt{a^2-r^2}$

$u = a^2 - r^2$

$du = -2r\,dr$

$\Rightarrow \iint_S F \cdot \hat{n}\,d\sigma = \iint_R \frac{F \cdot \nabla f}{|\nabla f \cdot p|}\,dA = \int_0^{2\pi}\int_0^{\frac{a}{\sqrt{2}}} \frac{\cancel{2}a^2}{\cancel{2}\sqrt{a^2-r^2}}\,r\,dr\,d\theta$

$= -\frac{a^2}{2}\int_0^{2\pi}\int_{r=0}^{r=\frac{a}{\sqrt{2}}} \frac{du}{\sqrt{u}}\,d\theta = -\frac{a^2}{2}\int_0^{2\pi} \frac{(a^2-r^2)^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_{r=0}^{r=\frac{a}{\sqrt{2}}} d\theta$

$= -a^2\left[\left(\frac{a^2}{2}\right)^{\frac{1}{2}} - (a^2)^{\frac{1}{2}}\right](2\pi)$

$= 2\pi a^2\left(a - \frac{a}{\sqrt{2}}\right) = \boxed{2\pi a^3\left(1 - \frac{1}{\sqrt{2}}\right)}$

(b) $\boxed{\text{Divergence!}}$

$\nabla \cdot F = 1+1+1 = 3$

$\iint_S F \cdot \hat{n}\,d\sigma = \iiint_V \nabla \cdot F\,dV = \int_0^{2\pi}\int_0^a\int_0^{\frac{\pi}{4}} 3\rho^2 \sin\phi\,d\phi\,d\rho\,d\theta$

$= 3\left(\int_0^{\frac{\pi}{4}} \sin\phi\,d\phi\right)\left(\int_0^a \rho^2\,d\rho\right)\left(\int_0^{2\pi} d\theta\right)$

$= 3\left(-\cos\phi\Big|_0^{\frac{\pi}{4}}\right)\left(\frac{\rho^3}{3}\Big|_0^a\right)(2\pi) = 2\pi a^3\left(-\frac{1}{\sqrt{2}} + 1\right)$

$= \boxed{2\pi a^3\left(1 - \frac{1}{\sqrt{2}}\right)}$

(5)