INSTRUCTIONS: Write your name and your instructor's name on the front of your work. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

Note: this is a Calculus III final exam. While some problems may be solved using techniques from Calculus I or II, you may not receive full credit if you do so, even if the final result is correct.

1. (40 Points) Consider the function $f(x, y) = xy^2 - xy$.

   (a) Locate, classify, and evaluate the local extreme values of $f(x, y)$.

   (b) If one starts at the point $(1, 1)$ and moves a distance $\Delta s = 0.1$ directly away from the origin, estimate (using Calculus III concepts) the change in the value of f.

   (c) Estimate the error in your estimate of Δf from part (c).

2. (40 points) The Grinch likes to keep his room dark. Unfortunately, he has a window in the shape of a half–circle of radius R that just lets in too much light. Specifically, the shape of the window is the region described by $x^2 + y^2 \leq R^2$ for $y \geq 0$. The Grinch plans to place a rectangular piece of plywood over as much of the glass as possible. The plywood is a wide and b tall. See the sketch above.

   (a) In terms of R, determine the optimum values of a and b so The Grinch can block out as much light as possible.

   (b) In terms or R, what is the maximum surface area of the window that The Grinch can block out?

3. (40 points) You need to evaluate the integral $\int_{y=0}^{2/3} \int_{x=y}^{2-2y} (x+2y)(x-y)^2 \, dx \, dy$, so you consider using the transformation $u = x + 2y$ and $v = x - y$.

   (a) Invert the transformation and clearly write x and y in terms of u and v.

   (b) Clearly show the regions of integration in both the x-y and u-v planes.

   (c) Rewrite the original integral in terms the new variables u and v.

   (d) Evaluate the new integral.

4. (40 Points) Consider the vector field $\mathbf{F} = (2xyz + 5z)\,\mathbf{i} + e^x \cos(yz)\,\mathbf{j} + x^2 y\,\mathbf{k}$.

(a) Determine if $\mathbf{F}$ is a conservative vector field.

(b) Let S denote the surface described by $y = 10 - x^2 - z^2$ for $y \geq 1$. Determine the value of $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$, where $\mathbf{n}$ is the rightward pointing normal to the surface S. Be sure to state any theorems from Calculus III if you use them.

5. (40 Points) Consider the integral $V = \displaystyle\int_{\theta=0}^{2\pi} \int_{\rho=0}^{a} \int_{\phi=0}^{\pi/4} \rho^2 \sin(\phi)\ d\phi\ d\rho\ d\theta$ that determines the volume of an object, and the vector field given by $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

(a) Calculate the total outward flux of the field $\mathbf{F}$ over the entire surface of the object. Be sure to clearly indicate the flux over each portion of the surface.

(b) If possible, verify your calculation in part (a) using any theorem(s) from Calculus III, and clearly state your reasoning! Otherwise, clearly write "Cannot be verified."

6. (40 Points) Consider $f(x, y) = (x-2)^2 + (y-3)^4$ (yes, that's a 4 up there) for $0 < x$ and $0 < y$.

(a) Determine the location and value of all extreme values of f. Be sure to classify your extreme values of f.

(b) Consider the extreme value of f closest to the origin $(0, 0)$. At that location, determine the second–order Taylor approximation to $f(x, y)$.

(c) Estimate the error associated with using your second–order Taylor approximation from part (b) if you go no further than 0.1 units away in any direction from the center of the expansion. (You may leave your result in terms of factorials, simple decimals, powers, etc.)

7. (40 points) A piece of wire of length L can be cut into two pieces. One piece is bent to form a square, while the remaining length is bent to form an equilateral triangle. Let S be the length of one side of the square, and let T be the length of one side of the triangle. Determine the values of S and T, in terms of L, that enclose the maximum total area inside the square and the triangle.

   Although this problem can be worked using Calculus I concepts, you need to work this problem using Calculus III principles. Clearly explain your approach as you work through the problem. Warning: no explanation might lead to no points! (Hint: you may find it useful to graph, in the S-T plane, the constraint curve and a few level curves of the objective function.)

8. (40 Points) Two small insects, Holly and Hank, are both moving along the path $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + (t + t^2)\,\mathbf{k}$, for $t \geq 0$, which is actually on the plane $z = x + y$. As they travel, they also move through a temperature field $T(x, y, z) = x^2y^2 + z^2 + 2$. At time $t = 1$, Hank says something really insensitive and Holly and Hank suddenly move off in different directions. Holly moves off in the direction of the local unit bi-normal to the path, $\mathbf{B}$, and Hank moves off in the direction of the greatest increase in temperature.

(a) Determine the unit vector for Holly's new direction.

(b) Determine the unit vector for Hank's new direction.

(c) If Holly goes a distance of 0.1 units in her new direction, estimate the temperature change that she experiences.

(d) Calculate Hank's speed when he changes directions.

(e) After changing directions, if Hank continues to travel at the same speed you calculated in part (d) for a time 0.1, estimate the temperature change that he experiences.

(f) Bonus points: Are Hank and Holly still on the original plane. If so, how do you know this? If not, how far are they from the plane?

Post script:

Holly, clearly being the smarter of the two, later went on to take APPM 3350 Advanced Engineering Calculus and then APPM 4380 Modeling in Applied Mathematics. She got an A in both classes. She is now a non-traditional student working on her PhD at CU Boulder. You can often times see her creeping along the floor right next to the walls in the Engineering Building. She wears a safety–green vest for increased visibility. Watch for her and tell her that Adam says 'Hi.'

Hank went to mandatory sensitivity training, but was eventually removed from the class by the instructor. Eventually he was stepped on in the Autumn of his years.

9. (40 Points) A small insect named Herman flies through the air along the path $\mathbf{r}(t) = \mathbf{i}+t^2\,\mathbf{j}+(1-t^2)\,\mathbf{k}$ for $0 \leq t \leq 1$. Two separate forces act on Herman as he flies along his merry way. The first is the drag force due to wind resistance, $\mathbf{F_1}$, and the second is a force field, $\mathbf{F_2}$, that depends on Herman's location in space, as defined below in part (c). You will calculate the work (flow) due to each of these forces.

(a) The drag force acting on Herman is opposite the direction of his motion and is directly proportional to his speed, $|\mathbf{v}|$. Hence, $\mathbf{F_1} = -|\mathbf{v}|\mathbf{T}$. Calculate the work (flow) due to $\mathbf{F_1}$ along Herman's path for $0 \leq t \leq 1$.

(b) If your calculation in part (a) can be verified using an alternate calculation, then do so, and clearly explain your reasoning. Otherwise, write "Cannot be verified."

(c) While Herman flies along his path $\mathbf{r}(t)$, he also moves through another force field, $\mathbf{F_2} = x^2\,\mathbf{i} + 2yz\,\mathbf{j} + y^2\,\mathbf{k}$. Calculate the work (flow) due to $\mathbf{F_2}$ along Herman's path for $0 \leq t \leq 1$.

(d) If your calculation in part (c) can be verified using an alternate calculation, then do so, and clearly explain your reasoning. Otherwise, write "Cannot be verified."

10. (40 Points) Consider the finite object bounded on top by the surface $z = 1$, on the bottom by $z = 0$, and the side by $x^2 + y^2 + z^2 = 2$.

   (a) Calculate the outward flux of the vector field $\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j} + xyz\,\mathbf{k}$ over the entire surface of the object. Be sure to clearly identify the flux over each part of the bounding surface.

   (b) If possible, verify your calculation in part (a) using any theorem(s) from Calculus III, and clearly state your reasoning! Otherwise, write "Cannot be verified."

   (c) Calculate the circulation along the counter–clockwise path around the "top edge" of the object.

   (d) If possible, verify your calculation in part (c) using any theorem(s) from Calculus III, and clearly state your reasoning! Otherwise, write "Cannot be verified."

ENJOY YOUR BREAK AND STAY SAFE!