(1) 

(a)

$f(x,y) = (x-2)^2 + (y-3)^4$

$f_x = 2(x-2)$ $\qquad f_x = 0 \Rightarrow x = 2 \qquad (2,3)$

$f_y = 4(y-3)^3$ $\qquad f_y = 0 \Rightarrow y = 3$

$f_{xx} = 2$

$f_{yy} = 12(y-3)^2 \qquad D = f_{xx}f_{yy} - f_{xy}^2 = 2 \cdot (y-3)$

$f_{xy} = 0 \qquad D(2,3) = 0$

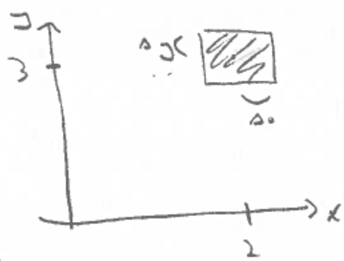
but it's actually a min (global)

By inspection, (0,0) is a local max, but no global max exists

(4) $P_2(2,3) = f(2,3) + \left(f_x(2,3)(x-2) + f_y(2,3)(y-3)\right) + \frac{1}{2}\left(f_{xx}(2,3)(x-2)^2 + 2f_{xy}(2,3)(x-2)(y-3) + f_{yy}(2,3)(y-3)^2\right)$

$= 0 + \left(0(x-2) + 0(y-3)\right) + \frac{1}{2}\left(2(x-2)^2 + 2(0)(x-2)(y-3) + 0(y\right.$

$= \boxed{x-2}$



(5) use $\Delta x, \Delta y = 0.1 \Rightarrow$

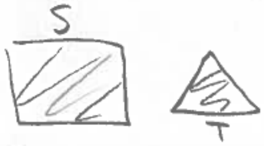
$$\left.\begin{aligned} f_{xxx} &= 0 \\ f_{xxy} &= 0 \\ f_{xyy} &= 0 \\ f_{yyy} &= 24(y-3) \end{aligned}\right\} \text{Max } \left|f_{yyy}\right|\Big|_{y=3.1} = 24(3.1-3) = 0.1(24) = 2.4 = M$$

$$\Rightarrow \text{Error} \le \frac{M}{3!}(\Delta x + \Delta y)^3 = \frac{2.4}{6}(0.2)^2$$

$$= \boxed{\frac{2.4}{3}(0.04)}$$

(1)

②



Max Area under $4S+3T=L$

Note: Area of triangle $\Rightarrow \frac{\sqrt{3}}{4}T^2$

$\Rightarrow$ Maximize total Area $= \frac{\sqrt{3}}{4}T^2 + S^2$

$\Rightarrow f(S,T) = S^2 + \frac{\sqrt{3}}{4}T^2$

$g(S,T) = 4S+3T$

$$\boxed{\begin{aligned} \nabla f &= \langle 2S, \tfrac{\sqrt{3}}{2}T\rangle \\ \nabla g &= \langle 4, 3\rangle \\ 4S+3T &= L \end{aligned}}$$

$$\Rightarrow \boxed{\begin{aligned} 2S &= \lambda 4 \\ \tfrac{\sqrt{3}}{2}T &= \lambda 3 \\ 4S+3T &= L \end{aligned}} \longrightarrow \frac{S}{2} = \lambda \Rightarrow \frac{\sqrt{3}}{\cancel{2}}T = \frac{S}{\cancel{2}}3 \Rightarrow T = \sqrt{3}S$$

$$\Rightarrow 4S + 3\sqrt{3}S = L \Rightarrow \quad S = \frac{L}{4+3\sqrt{3}}$$

$$T = \frac{\sqrt{3}L}{4+3\sqrt{3}}$$

$$\left(\text{Check: } 4S+3T = \frac{4L}{4+3\sqrt{3}} + \frac{3\sqrt{3}L}{4+3\sqrt{3}} = \frac{(4+3\sqrt{3})L}{4+3\sqrt{3}} = L \checkmark\right)$$

$$\Rightarrow \text{Area} = f\left(\frac{L}{4+3\sqrt{3}}, \frac{\sqrt{3}L}{4+3\sqrt{3}}\right) = \left(\frac{L^2}{(4+3\sqrt{3})^2}\right) + \frac{\sqrt{3}}{4}\left(\frac{3L^2}{(4+3\sqrt{3})^2}\right)$$

$$= \frac{4L^2 + \sqrt{3}\,3L^2}{4(4+3\sqrt{3})^2} = \frac{(4+\sqrt{3}\,3)L^2}{4(4+3\sqrt{3})^2} = \frac{L^2}{4(4+3\sqrt{3})}$$

Check end points: $S=0 \Rightarrow 3T = L \Rightarrow T = \frac{L}{3}$

$$\Rightarrow f(0, \tfrac{L}{3}) = \frac{\sqrt{3}}{4}\frac{L^2}{9}$$

$$T=0 \Rightarrow 4S = L \Rightarrow S = \frac{L}{4}$$

$$f(\tfrac{L}{4}, 0) = \frac{L^2}{16} \leftarrow \text{Max}$$

Max at $S = \frac{L}{4}, T=0$! $\Rightarrow$ Area $= \frac{L^2}{16}$

Actually a min!

③

(3) (a) Holly's new direction: binormal

Since $\vec{r}(t)$ is on the plane with normal $\langle 1,1,-1\rangle$,

$$B \parallel \hat{n} \Rightarrow B\text{normal} = \boxed{\frac{\langle 1,1,-1\rangle}{\sqrt{3}}}$$

(b) greatest increase: $\nabla T(\vec{r}(1))$

$$\nabla T = \langle 2xy^2, 2x^2y, 2z\rangle \qquad \vec{r}(1) = \langle 1,1,2\rangle$$

$\nabla T(1,1,2) = \langle 2,2,4\rangle \Rightarrow$ Heath's new direction is

$$\frac{\langle 2,2,4\rangle}{\sqrt{4+4+16}} = \frac{\langle 1,1,2\rangle}{\sqrt{1+1+4}} = \boxed{\frac{\langle 1,1,2\rangle}{\sqrt{6}}}$$

(c) $$\frac{dT}{ds} = \nabla T\cdot\vec{u} \Rightarrow \Delta T \approx (\nabla T\cdot\vec{u})\Delta s$$

$$t=1 \Rightarrow \vec{r}(1) = \langle 1,1,2\rangle \Rightarrow \Delta T \approx \langle 2,2,4\rangle\cdot\frac{\langle 1,1,-1\rangle}{\sqrt{3}}(0.1)$$

$$= 0.1\frac{(2+2-4)}{\sqrt{3}} = \boxed{0}$$

(d) $$\text{speed} = |\vec{r}'(t)| = |\langle 1, 2t, 1+2t\rangle| = \sqrt{1+4t^2+(1+2t)^2}$$

$$= \sqrt{1+4t^2+1+4t^2+4t} = \sqrt{2+4t+8t^2}$$

$$\text{speed}\Big|_{t=1} = \sqrt{2+4+8} = \boxed{\sqrt{14}}$$

(e) $\Rightarrow ds = \sqrt{14}\,dt$

$$\Rightarrow \frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt} = \frac{ds}{dt}(\nabla T\cdot\vec{u}) \Rightarrow \Delta T \approx (\nabla T\cdot\vec{u})\frac{ds}{dt}\Delta t$$

$$t=1 \Rightarrow \Delta T \approx \left(\langle 2,2,4\rangle\cdot\frac{\langle 1,1,2\rangle}{\sqrt{6}}\right)\sqrt{14}\,(0.1)$$

$$= (2+2+8)\frac{\sqrt{14}}{\sqrt{6}}(0.1) = \boxed{\frac{\sqrt{7}}{\sqrt{3}}\,12(0.1)}$$

3

(4) $\vec{r}(t) = \langle 1, t^2, 1-t^2 \rangle$

(a) $v(t) = \vec{r}'(t) = \langle 0, 2t, -2t \rangle \quad |v(t)| = \sqrt{4t^2+4t^2} = 2\sqrt{2}\,t$

$$\vec{T}(t) = \frac{v(t)}{|v(t)|} = \frac{\langle 0, 2t, -2t \rangle}{2\sqrt{2}\,t} = \frac{\langle 0, 1, -1 \rangle}{\sqrt{2}}$$

$$\Rightarrow \vec{F}_1 = -|v(t)|\,\vec{T} = -2\sqrt{2}\,t\,\frac{\langle 0,1,-1\rangle}{\sqrt{2}} = \langle 0, -2t, 2t \rangle$$

$$\text{Flow} = \int \vec{F}_1 \cdot \vec{T}\, ds = \int_0^1 \vec{F} \cdot \vec{r}'(t)\, dt = \int_0^1 \langle 0, -2t, 2t \rangle \cdot \langle 0, 2t, -2t \rangle$$
$$= \int_0^1 -4t^2 - 4t^2\, dt = -8\,\frac{t^3}{3}\Big|_0^1 = \boxed{-\frac{8}{3}}$$

(b) Cannot be verified

(c) $\vec{F}_2 = \langle x^2, 2yz, y^2 \rangle = \langle 1, 2t^2(1-t^2), t^4 \rangle$
$$= \langle 1, 2t^2 - 2t^4, t^4 \rangle$$

$$\text{Flow} = \int_0^1 \vec{F} \cdot \vec{T}\, ds = \int_0^1 \vec{F} \cdot \vec{r}'(t)\, dt$$
$$= \int_0^1 \langle 1, 2t^2 - 2t^4, t^4 \rangle \cdot \langle 0, 2t, -2t \rangle\, dt$$
$$= \int_0^1 4t^3 - 4t^5 - 2t^5\, dt = \cancel{4}\frac{t^4}{\cancel{4}} - \cancel{6}\frac{t^6}{\cancel{6}}\Big|_0^1 = \boxed{0}$$

(d) $$\nabla \times F_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & 2yz & y^2 \end{vmatrix} = \vec{i}(2y - 2y) - \vec{j}(0-0) + \vec{k}(0-0)$$
$$= \vec{0} \Rightarrow \text{conservative}$$

Yes, we can use Fundamental Theorem of Line Integrals!

$F = \nabla f \Rightarrow f_x = x^2 \Rightarrow f = \int x^2\,dx = \frac{x^3}{3} + g_1(y,z)$

$f_y = 2yz \Rightarrow f = \int 2yz\,dy = y^2 z + g_2(x,z)$

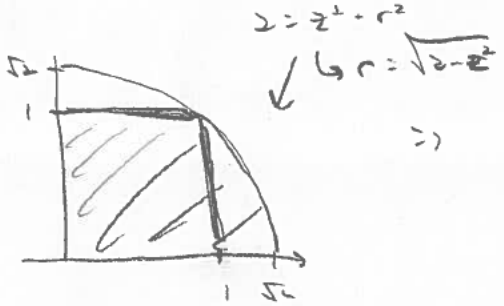
$f_z = y^2 \Rightarrow f = \int y^2\,dz = y^2 z + g_3(x,y)$

$$\Rightarrow f(x,y,z) = \frac{x^3}{3} + y^2 z$$

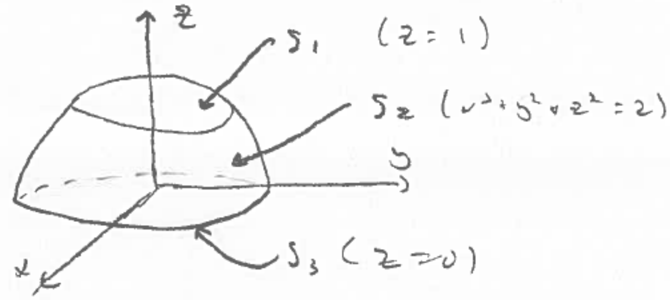
$$\text{Flow} = f(\vec{r}(1)) - f(\vec{r}(0)) = f(1,1,0) - f(1,0,1)$$
$$= \left(\frac{1}{3} + 0\right) - \left(\frac{1}{3} + 0\right) = \boxed{0} \quad \checkmark$$

(4)

(5)



$\Rightarrow$



(a) $S_1 : f_1 = z = 1 \Rightarrow \nabla f_1 = \langle 0, 0, 1 \rangle$ $\qquad F = \langle y, -x, xyz \rangle$

$S_2 : f_2 = x^2 + y^2 + z^2 \Rightarrow \nabla f_2 = \langle 2x, 2y, 2z \rangle$

$S_3 : f_3 = -z = 0 \Rightarrow \nabla f_3 = \langle 0, 0, -1 \rangle$

project each surface onto xy-plane : $S_1 : x^2 + y^2 \le 1$

$S_2 : 1 \le x^2 + y^2 \le 2$, $\vec{p} = \vec{k}$

$S_3 : x^2 + y^2 \le 2$

$$\text{Flux} = \iint_{S_1} F \cdot \vec{n}\, d\sigma + \iint_{S_2} F \cdot \vec{n}\, d\sigma + \iint_{S_3} F \cdot \vec{n}\, d\sigma$$

$$= \iint_{R_1} \frac{xy\overbrace{z}^{z=1}}{1}\, dA + \iint_{R_2} \frac{\cancel{2xy} - \cancel{2xy} + 2xyz^2}{2z}\, dA + \iint_{R_3} \frac{-xy\overbrace{z}^{z=0}}{1}\, dA$$

$$= \iint_{R_1} xy\, dA + \iint_{R_2} xy\underbrace{z}_{z = \sqrt{2 - x^2 - y^2}}\, dA + \underbrace{\iint_{R_3} 0\, dA}_{0}$$

$$= \int_0^{2\pi}\int_0^1 r^2 \cos\theta \sin\theta\, r\, dr\, d\theta + \int_0^{2\pi}\int_1^{\sqrt{2}} r^2 \cos\theta \sin\theta \sqrt{1 - r^2}\, r\, dr\, d\theta$$

$$= \left(\int_0^1 r^3\, dr\right)\left(\int_0^{2\pi} \cos\theta \sin\theta\, d\theta\right) + \left(\int_1^{\sqrt{2}} r^3 \sqrt{1 - r^2}\, dr\right)\left(\int_0^{2\pi} \cos\theta \sin\theta\, d\theta\right)$$

$$= \left(\frac{r^4}{4}\Big|_0^1\right)\left(\frac{\sin^2\theta}{2}\Big|_0^{2\pi}\right) + \left(\quad '' \quad \right)\left(\frac{\sin^2\theta}{2}\Big|_0^{2\pi}\right)$$

$$= \frac{1}{4}(0) + \left('' \right)(0) = \boxed{0}$$

(b) Divergence : $\nabla \cdot F = 0 + 0 + xy = xy = r^2 \cos\theta \sin\theta$

$$\Rightarrow \text{Flux} = \iiint_V \nabla \cdot F\, dV = \int_0^{2\pi}\int_0^1\int_0^{\sqrt{2 - z^2}} r^2 \cos\theta \sin\theta\, r\, dr\, dz\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{r^4}{4} \cos\theta \sin\theta \Big|_0^{\sqrt{2 - z^2}} dz\, d\theta = \frac{1}{4}\int_0^{2\pi}\int_0^1 (2 - z^2)^2 \overbrace{}^{\cos\theta \sin\theta} dz\, d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}\int_0^1 (4 - 4z^2 + z^4) \cos\theta \sin\theta\, dz\, d\theta = \frac{1}{4}\int_0^{2\pi}\left(4z - \frac{4z^3}{3} + \frac{z^5}{5}\Big|_0^1\right) \cos\theta \sin\theta\, d\theta$$

$$= \frac{1}{4}\left(4 - \frac{4}{3} + \frac{1}{5}\right)\int_0^{2\pi} \cos\theta \sin\theta\, d\theta$$

$$= \left(1 - \frac{1}{3} + \frac{1}{20}\right) \frac{\sin^2\theta}{2}\Big|_0^{2\pi} = \left(1 - \frac{1}{3} + \frac{1}{20}\right)(0) = \boxed{0}\ ✓$$

(5)

(c) $z = 1 \Rightarrow x^2 + y^2 + 1 = 2 \Rightarrow \boxed{x^2 + y^2 = 1,\ z = 1}$

$\Rightarrow \vec{r}(t) = \langle \cos t, \sin t, 1 \rangle \qquad F = \langle y, -x, xyz \rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 0 \rangle \qquad = \langle \sin t, -\cos t, \cos t \sin t \rangle$

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \int_0^{2\pi} \vec{F} \cdot \vec{r}\,'(t)\, dt = \int_0^{2\pi} -\sin^2 t - \cos^2 t \, dt$$

$$= -\int_0^{2\pi} dt = \boxed{-2\pi}$$

(d) in $z = 0$ plane $\Rightarrow$ Green's!

$R =$ 

$\Rightarrow$ Flux $= \iint_R Q_x - P_y \, dA$

$$= \int_0^{2\pi} \int_0^1 (-1 - 1)\, r \, dr \, d\theta$$

$$= -2 \left(\int_0^1 r\, dr \right) \left(\int_0^{2\pi} d\theta \right)$$

$$= -2 \left(\frac{r^2}{2} \Big|_0^1 \right) (2\pi) = \boxed{-2\pi} \ \checkmark$$

5